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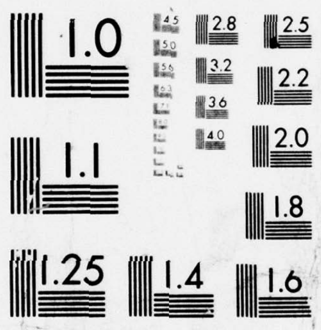

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The Conditional Probability of Random Harmonic Sets

D. A. Swick

Advanced Projects Group
Acoustics Division

June 1978

NAVAL RESEARCH LABORATORY
Washington, D.C.

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The approximate conditional probability that m random variables out of a set of n uniformly and independently distributed on (a,b) are integral multiples of a "fundamental" to within a tolerance of 100t%, given that the "fundamental" is an integral divisor of the smallest of the m to within the same tolerance has been calculated. The algorithm has been implemented in an efficient Fortran subroutine.

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CONTENTS

THE CONDITIONAL PROBABILITY OF RANDOM HARMONIC SETS

Introduction

The approximate probability that m random variables out of a set of n uniformly and independently distributed on (a,b) are harmonically related to within a tolerance of 100t% has been calculated in [1]. If accidental harmonic multiples are to be avoided, the need for high accuracy in frequency measurements is indicated by these results. Application of them has revealed both a computational and a conceptual problem which are the subjects of this paper. The computational problem is discussed in connection with Eq. 2, below. A key element in its elimination is a by-product of the solution of the conceptual problem: The derivation in [1] is valid when the smallest member of a "harmonically related" subset is a low-order harmonic. (It is compared there with the results of a small-scale simulation for the cases where the smallest member is the fundamental or the second harmonic.) If the smallest member is a harmonic of much higher order, however, asking a posteriori for the probabilities of multiples of the indicated fundamental may be "data-snooping", and too dependent on the way the data fall out. It is more appropriate to ask for the <u>conditional</u> probability that all elements of a subset be multiples of a fundamental, given that the smallest member is such a multiple. Before addressing this problem, the results of [1] will be summarized to establish the notation and to disclose the computational difficulty.

Unconditional Probability: Summary of Results

We consider a set of n random variables represented by $X_1,\ldots,X_n$, independently and uniformly distributed on (a,b), and an ordered subset of them, $Y_1 \le Y_2 \le \ldots \le Y_m$, where $m=2,3,\ldots,n$. If the Y_i are all multiples of Y_1/k_1, to within a tolerance of 100t%, where k_1 is the harmonic number of the smallest member of the subset, we say that they form a <u>chain</u> of harmonics of length m. The probability of this is given by

$$P_m(k_1,k_2,\ldots,k_m)$$

$$= \Pr\{(Y_1,\ldots,Y_m):(1-t)k_iY_1/k_1 \le Y_i \le (1+t)k_iY_1/k_1, i=2,\ldots,m\}$$

$$\approx \frac{(m-1)!}{(b-a)^m}\left(\frac{2t}{k_1k_m}\right)^{m-1}\left(\prod_{i=2}^{m-1} k_i\right)\left[(k_1b)^m-(k_ma)^m\right] \qquad (1)$$

where $k_1 < k_2 < \cdots < k_m$, and terms of $O(t^2)$ have been dropped.

Note: Manuscript submitted May 15, 1978.

If $L = [bk_1/a(1+t)]$ is the largest harmonic permitted by the range and tolerance, then the probability that a particular subset forms a chain of length m for some set of k_i is

$$P_m(k_1) = \sum_{k_2=k_1+1}^{L-m+2} \sum_{k_3=k_2+1}^{L-m+3} \cdots \sum_{k_m=k_{m-1}+1}^{L} p_m(k_1,\ldots,k_m). \quad (2)$$

The approximate probability of at least one chain of length m among the n random variables is given by

$$P(n,m,k_1) = \sum_{r=1}^{M} (-1)^{r+1}\binom{M}{r}\left[P_m(k_1)\right]^r , \quad (3)$$

where $M = \binom{n}{m}$.

The computational problem implicit in (3) is discussed in [1] and is resolved by a simple recursion formula. A computational problem not resolved there, however, is that of Eq. 2. For a "reasonable" interval (a,b), L can be large, and the straightforward computation of the nested sums can become economically prohibitive for m larger than five or six. Even for small m, repeated calculations can be costly; they are not feasible for chains of ten or more.

A partial solution to this difficulty is obtained by reformulating the problem in terms of differences between successive variables rather than multiples of a common "fundamental". Since successive differences can be the same, unlike successive multiples, it is possible to effect a computational advantage in an analog of (2) by combining terms. This technique shifts the computational problem to longer chains, but it does not solve it. It is eliminated in the conditional formulation which follows.

<u>Conditional Probability of Harmonic Chains</u>

As before, let $Y_1 \le Y_2 \le \ldots \le Y_m$ be an ordered subset of the n random variables $X_1, X_2, \ldots, X_n$ which are independently and uniformly distributed on (a,b). The conditional probability of a chain of harmonics of length m, i.e., the conditional probability that all Y_i are multiples of Y_1/k_1 to within a tolerance of 100t%, given that $f_o = Y_1/k_1$ to within the same tolerance is given by

$$p_m(k_2,\ldots,k_m \mid k_1)$$

$$= \Pr\Big[\{(Y_1,\ldots,Y_m):(1-t)k_iY_1/k_1 \le Y_i \le (1+t)k_iY_1/k_1; i=2,\ldots,m\}$$

$$\cap \{(Y_1):(1-t)k_1f_o \le Y_1 \le (1+t)k_1f_o\}\Big]$$

$$\cdot \Big[\Pr\{(Y_1):(1-t)k_1f_o \le Y_1 \le (1+t)k_1f_o\}\Big]^{-1}$$

$$= \frac{\dfrac{m!}{(b-a)^m}\displaystyle\int_{(1-t)k_1f_o}^{(1+t)k_1f_o}\prod_{i=2}^{m}\left\{\int_{(1-t)k_iy_1/k_1}^{(1+t)k_iy_1/k_1} dy_i\right\} dy_1}{\dfrac{1}{b-a}\displaystyle\int_{(1-t)k_1f_o}^{(1+t)k_1f_o} dy}$$

$$= \qquad m!\left(\frac{2tf_o}{b-a}\right)^{m-1}\prod_{i=2}^{m} k_i\,, \tag{4}$$

where $k_1 < k_2 < \cdots < k_m$, and $a/(1-t) \le k_if_o \le b/(1+2t)\ \forall i$, neglecting terms of $O(t^2)$.

Now let $L = [b/f_o(1+t)^2]$ be the largest possible harmonic. The conditional probability that a particular subset forms a chain of length m becomes

$$P_m(k_1) = m!\left(\frac{2tf_o}{b-a}\right)^{m-1}\sum_{k_2=k_1+1}^{L-m+2}\ \sum_{k_3=k_2+1}^{L-m+3}\cdots\sum_{k_m=k_{m-1}+1}^{L}\ \prod_{i=2}^{m} k_i\ . \tag{5}$$

The simplicity of (4), as contrasted with (1), permits an analytic evaluation of (5), thus avoiding the computational problem of (2). The approximate conditional probability of at least one chain of

length m among the n random variables is then given by (3).

Writing (5) as a sequence of nested sums, we have

$$P_m(k_1) = m! \left(\frac{2tf_o}{b-a}\right)^{m-1} S_2 \tag{6}$$

where

$$S_2 = \sum_{k_2=k_1+1}^{L-m+2} S_3,\ S_3 = \sum_{k_3=k_2+1}^{L-m+3} S_4, \ldots, S_{m-1} = \sum_{k_{m-1}=k_{m-2}+1}^{L-1} S_m \ . \tag{7}$$

The innermost sum is

$$S_m = \left(\prod_{i=2}^{m-1} k_i\right) [L(L+1)-k_{m-1}-k_{m-1}^2]/2 \ . \tag{8}$$

Either directly from (5), or from (6) and (8), we have

$$P_2(k_1) = \frac{2tf_o}{b-a} [L(L+1)-k_1-k_1^{\,2}] \ .$$

For $m > 2$, we write the general term in (7) as

$$S_{m-i} = \sum_{k_{m-i}=k_{m-i-1}+1}^{L-i} S_{m-i+1}; i=1,\ldots,m-2 \ . \tag{9}$$

To evaluate this sequence of sums we assume a relationship of the form

$$S_{m-i+1} = \left(\prod_{\alpha=2}^{m-i} k_\alpha\right) \sum_{\beta=1}^{2i+1} A_{i\beta}\left(k_{m-i}\right)^{\beta-1};\ i=1,\ldots,m-1 \ . \tag{10}$$

For $i = m-1$ this becomes

$$S_2 = \sum_{\beta=1}^{2m-1} A_{m-1,\beta}\, k_1^{\,\beta-1} \ ;\ m=2,3,\ldots, \tag{11}$$

from which (6) can be evaluated once the coefficients $A_{m-1,\beta}$ have been determined. For i=1, (10) becomes

$$S_m = \left(\prod_{\alpha=2}^{m-1} k_\alpha\right)\left[A_{11} + A_{12}k_{m-1} + A_{13}k_{m-1}^{\ 2}\right].$$

Comparison with (8) yields

$$A_{11} = L(L+1)/2,\ A_{12} = A_{13} = -1/2, \tag{12}$$

valid for all m.

Substituting (10) in (9) and letting $k = k_{m-i}$ we have

$$S_{m-i} = \left(\prod_{\alpha=2}^{m-i-1} k_\alpha\right) \sum_{k=k_{m-i-1}+1}^{L-i} \sum_{\beta=1}^{2i+1} A_{i\beta}k^\beta$$

$$= \left(\prod_{\alpha=2}^{m-i-1} k_\alpha\right) \sum_{\beta=1}^{2i+1} A_{i\beta}\left[\sum_{k=1}^{L-i} k^\beta - \sum_{k=1}^{k_{m-i-1}} k^\beta\right]. \tag{13}$$

In terms of the Bernoulli polynomials $B_n(x) = \sum_{k=o}^{n} b_k^{(n)} x^k$ [2, Chap. 23] we can write

$$\sum_{k=1}^{s} k^\beta = \frac{B_{\beta+1}(s+1) - B_{\beta+1}(0)}{\beta+1} = \frac{1}{\beta+1}\sum_{\gamma=1}^{\beta+1} b_\gamma^{(\beta+1)}(s+1)^\gamma,$$

so that (13) becomes

$$S_{m-i} = \left(\prod_{\alpha=2}^{m-i-1} k_\alpha\right) \sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_\gamma^{(\beta+1)}[(L-i+1)^\gamma-(k_{m-i-1}+1)^\gamma]$$

$$= \left(\prod_{\alpha=2}^{m-i-1} k_\alpha\right) \sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_\gamma^{(\beta+1)}\left[(L-i+1)^\gamma - \sum_{\delta=1}^{\gamma+1} \binom{\gamma}{\delta-1}\left(k_{m-i-1}\right)^{\delta-1}\right]. \tag{14}$$

We collect powers of k_{m-i-1}. For $\delta=1$ we identify the coefficient of $(k_{m-i-1})^{0} = 1$ as

$$A_{i+1,1} = \sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_{\gamma}^{(\beta+1)} [(L-i+1)^{\gamma}-1]; \tag{15}$$

otherwise the coefficient of $(k_{m-i-1})^{\delta-1}$ is

$$A_{i+1,\delta} = -\sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} \binom{\gamma}{\delta-1} b_{\gamma}^{(\beta+1)}, \quad \delta=2,\ldots,2i+3, \tag{16}$$

both for $i=1,\ldots,m-2$. Using these coefficients, (14) becomes

$$S_{m-i} = \left(\prod_{\alpha=2}^{m-i-1} k_{\alpha}\right) \sum_{\delta=1}^{2i+3} A_{i+1,\delta}(k_{m-i-1})^{\delta-1}; \; i=0,1,\ldots,m-2,$$

which has the same form as (10) with i replaced by $i+1$. The values of $A_{1\delta}$ are given by (12). For $1 \le i \le m-2$ $A_{i+1,\delta}$ may be evaluated recursively using (15) and (16), with $A_{m-1,\delta}$, $\delta=1,\ldots,2m-1$, used to evaluate S_2 in (11). From (15)

$$A_{i+1,1} = \sum_{\beta=1}^{2i+1} d_{i\beta} A_{i\beta} \, ,$$

where

$$d_{i\beta} = \frac{1}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_{\gamma}^{(\beta+1)} [(L-i-1)^{\gamma}-1], \tag{17}$$

for $i=1,\ldots,m-2$ and $\beta=1,\ldots,2i+1$.

From (16),

$$A_{i+1,\delta} = \sum_{\beta=\epsilon}^{2i+1} C_{\beta\delta} A_{i\beta},$$

where

$$C_{\beta\delta} = -\frac{1}{\beta+1}\sum_{\gamma=1}^{\beta+1}\binom{\gamma}{\delta-1} b_\gamma^{(\beta+1)} \tag{18}$$

for $i=1,\ldots,m-2$, $\delta=2,\ldots,2i+3$, $\epsilon=\max(1,\delta-2)$, $\beta=\epsilon,\ldots,2i+1$, and $C_{\beta\delta} = 0$ if $\beta < \delta-2$. A storage advantage may be obtained from the observation that the $A_{i\delta}$ may be stored in a "lower triangular" two-dimensional array and the $C_{\beta\delta}$ may be stored in an "upper triangular" array. Thus they may share a single 2m by 2m-1 array.

The coefficients $b_k^{(n)}$ of the Bernoulli polynomials $B_n(x)$ are given in Table 23.1 of [2] for $0 \le n \le 15$. They may be obtained in general from

$$b_k^{(n)} = \binom{n}{k} B_{n-k},\ k=0,\ldots,n,$$

where $B_n=B_n(0)$, $B_o=1$, and [2]

$$\sum_{k=0}^{n-1}\binom{n}{k} B_k = 0.$$

The computation of the conditional probability of harmonic chains has been incorporated into a Fortran IV subroutine called PROHARM which is described and listed in the Appendix. The required binomial coefficients are obtained using subroutine BINO, also listed, as is CALLPRO, used to call the subroutines.

APPENDIX: Fortran implementation

Subroutine PROHARM, Figure 1, is a Fortran IV implementation of the algorithm for the computation of the conditional probability of harmonic chains. All parameters in the calling argument follow the notation of this paper. In principle there is no limit to the length of the chain; it is restricted here to $m \leq 20$ because of core limitations, with $n \geq m$ unrestricted. The arrays are dimensioned accordingly.

Terms of order t^2 were neglected in the text. Although they make no difference in the result, it is not much more difficult to calculate the exact coefficient as

$$(m-1)! \left(\frac{f_o}{b-a}\right)^{m-1} (2t)^{m-2}[(1+t)^m - (1-t)^m]$$

rather than the close approximation $m! \ (2tf_o/(b-a))^{m-1}$. This is done on lines 120-300 of PROHARM. L1 (line 310) is L+1. If m harmonics will not "fit" in the given range, Pr=0 is returned. Array AC is to be shared by $A_{i\delta}$ and $C_{\beta\delta}$, with the initial values given by (12) stored in row 22. Throughout the subroutine, the parameter MM is used to avoid repeating a previously-completed calculation; only those coefficients needed for M > MM are calculated.

The coefficients $b_k^{(n)}$ of the Bernoulli polynomials are calculated via lines 400-600, and are stored in array BB. Array CO stores the $d_{i\beta}$ which are calculated by lines 610-700 using (17). Equation 18 yields the $C_{\beta\delta}$, lines 720-840. These are stored in the upper right portion of array AC, while the lower left of AC holds the $A_{i\beta}$ which are calculated recursively, lines 850-950.

Equations 11 and 6 are implemented in lines 960-1010 and 1020, respectively, to obtain $P_m(k_1)$. Finally, the approximate probability of at least one chain, Eq. 3, is calculated as discussed in [1] via lines 1030-1170. This calculation usually converges rapidly, and is stopped when the next term changes the result by less than 0.0001%. When the probability is actually very close to unity, however, the finite word length of the computer may produce divergence in this calculation. In this case, Pr=1 is returned. The validity of this procedure has been established by observations of the changes in the probability introduced by small perturbations of the parameters.

A sample Fortran calling program, CALLPRO, is listed as Fig. 2. This program retrieves the subroutines PROHARM and BINO. The latter, used to calculate the binomial coefficients required by PROHARM, is listed as Fig. 3. Fig. 4 is a run of CALLPRO in which the conditional probability of at least one chain of length m, $2 \leq m \leq 20$, given

```
      SUBROUTINE PROHARM(A,B,N,M,T,F0,K1,PR)
      DIMENSION BB(38,38),AC(40,39),C0(18,37)
      M1=M-1
      M2=M-2
      TM=1.
      M1F=M1
      F=F1=F0/(B-A)
      TMM=TD=1.-T
      TMM=TMM*TD
      TPM=TP=1.+T
      TPM=TPM*TP
      IF(M.EQ.2)GOTO 2
      T2=2*T
      DO 1 K=1,M2
      MK=M1-K
      TM=TM*T2
      F=F*F1
      TMM=TMM*TD
      TPM=TPM*TP
 1    M1F=M1F*MK
 2    CC=M1F*TM*F*(TPM-TMM)
      L1=INT(B/(F0*TP*TP))+1
      IF(L1.GE.K1+M)GOTO 3
      PR=0.
      RETURN
 3    JM=2*M-1
      AC(22,1)=FLOAT(L1)*FLOAT(L1-1)/2.
      IF(M.LE.MM)GOTO 6
      AC(22,2)=AC(22,3)=-.5
      IF(M.EQ.2)GOTO 14
      MC=2*M1
      JM0=2*MM-1
      BB(1,1)=-.5
      I0=3
      IF(MM.NE.0)I0=2*(MM-2)+3
      DO 4 I=I0,MC
      I2=I-2
      C1=1.
      X=-1.
      DO 4 K=1,I2
      C1=C1*(I-K+1)/K
      X=X-C1*BB(1,K)
 4    BB(1,I-1)=X/I
      IF(MM.EQ.0)I0=2
      DO 5 I=I0,MC
      I1=I-1
      C1=1.
      BB(I,I)=1.
      DO 5 K=1,I1
      C1=C1*(I-K+1)/K
 5    BB(I,K)=C1*BB(1,I-K)
 6    DO 8 I=1,M2
      KM=2*I+1
      DO 8 K=1,KM
      KP=K+1
      X=0.
      FLI=FL0=FLOAT(L1-I)
      DO 7 L=1,KP
      X=X+BB(KP,L)*(FLI-1.)
 7    FLI=FLI*FL0
```

```
8     C0(I,K)=X/KP
      IF(M.LE.MM)GOTO 11
      KM=JM-2
      MM=M
      DO 10 J=2,JM
      J1=J-1
      K0=MAX0(1,J-2)
      IF(J.LE.JM0)K0=JM0-1
      DO 10 K=K0,KM
      LM=K+1
      X=0.
      DO 9 L=J1,LM
      CALL BINO(L,J1,IG)
9     X=X-IG*BB(LM,L)
10    AC(J-1,K)=X/LM
11    DO 13 I=2,M1
      KM=2*I-1
      JI=KM+2
      AC(I+21,1)=0.
      DO 12 K=1,KM
12    AC(I+21,1)=AC(I+21,1)+C0(I-1,K)*AC(I+20,K)
      DO 13 J=2,JI
      AC(I+21,J)=0.
      K0=MAX0(1,J-2)
      DO 13 K=K0,KM
13    AC(I+21,J)=AC(I+21,J)+AC(J-1,K)*AC(I+20,K)
14    F=1.
      S2=0.
      FL=FLOAT(K1)
      DO 15 J=1,JM
      S2=S2+AC(M1+21,J)*F
15    F=F*FL
      PM=CC*S2
      CALL BINO(N,M,NCM)
      KS=1
      S=PR=NCM*PM
      IF(NCM.EQ.1)RETURN
      DO 16 K=2,NCM
      KS=-1*KS
      S=((S*PM)/K)*(NCM-K+1)
      IF(S.LT.1.E-6*PR)RETURN
      IF(S.GT.1.E+12)GOTO 17
16    PR=PR+KS*S
      IF(PR.LT.0)PR=1.
      IF(PR.GT.1.)PR=1.
      RETURN
17    PR=1.
      RETURN
      END
```

FIG. 1 Subroutine PROHARM

```
      PROGRAM CALLPRO(INPUT,OUTPUT)
      PRINT,*RANGE: A,B*,
      READ,A,B
1     PRINT,*N,M,TOL,FØ,K1*,
      READ,N,M,T,FØ,K1
      CALL PROHARM(A,B,N,M,T,FØ,K1,PR)
      PRINT 2,PR
      GOTO  1
2     FORMAT(*  PR=*,E1Ø.2/)
      RETRIEVE(PROHARM)
      RETRIEVE(BINO)
      END
```

FIG. 2   Program CALLPRO

```
      SUBROUTINE BINO(N,M,K)
      L=N-M
      MM=M
      IF(L.GT.M)MM=L
      L=N-MM
      K=Ø
      IF(L.EQ.Ø)K=1
      IF(L.EQ.1)K=N
      IF(L.LE.1)RETURN
      K=L
      A=1.*N
      D=1.*K
      B=A/D
      DO 1 I=2,K
      A=A-1.
      D=D-1.
1     B=B*A/D
      K=INT(B+.5)
      RETURN
      END
```

FIG. 3 Subroutine BINO

```
PROGRAM  CALLPRO

RANGE: A,B ? .05,.8
N,M,TOL,F0,K1 ? 20,2,.003,.02,8
  PR=  1.00E+00

N,M,TOL,F0,K1 ? 20,4,.003,.02,8
  PR=  9.97E-01

N,M,TOL,F0,K1 ? 20,6,.003,.02,8
  PR=  9.77E-01

N,M,TOL,F0,K1 ? 20,7,.003,.02,8
  PR=  5.76E-01

N,M,TOL,F0,K1 ? 20,8,.003,.02,8
  PR=  1.38E-01

N,M,TOL,F0,K1 ? 20,9,.003,.02,8
  PR=  1.97E-02

N,M,TOL,F0,K1 ? 20,10,.003,.02,8
  PR=  2.06E-03

N,M,TOL,F0,K1 ? 20,11,.003,.02,8
  PR=  1.67E-04

N,M,TOL,F0,K1 ? 20,12,.003,.02,8
  PR=  1.05E-05

N,M,TOL,F0,K1 ? 20,13,.003,.02,8
  PR=  5.08E-07

N,M,TOL,F0,K1 ? 20,14,.003,.02,8
  PR=  1.88E-08

N,M,TOL,F0,K1 ? 20,15,.003,.02,8
  PR=  5.21E-10

N,M,TOL,F0,K1 ? 20,16,.003,.02,8
  PR=  1.06E-11

N,M,TOL,F0,K1 ? 20,17,.003,.02,8
  PR=  1.50E-13

N,M,TOL,F0,K1 ? 20,18,.003,.02,8
  PR=  2.93E-16

N,M,TOL,F0,K1 ? 20,19,.003,.02,8
  PR=  2.52E-19

N,M,TOL,F0,K1 ? 20,20,.003,.02,8
  PR=  7.23E-24
```

FIG. 4 $P(n,m,k_1)$ for $(a,b) = (.05,.8)$, $n=20$, $t=0.003$, $f_o=0.02$, $k_1=8$ and $2 \le m \le 20$.

f_o = 0.02 is calculated. A tolerance of 0.003, k_1=8, and n=20 random variables independently and uniformly distributed on (0.05, 0.8) are assumed. With these parameters and range of m, the probability drops from unity to essentially zero. The calculation shown in Fig. 4 took less than 6 CPU seconds, including compilation, on a CDC CYBER 174.

References

1. D. A. Swick, "Harmonic Relationships Among Random Variables," SIAM J. Appl. Math., 33, 3, pp. 490-498 (Nov. 1977).

2. M. Abramowitz and I. A. Stegun, Eds., "Handbook of Mathematical Functions," National Bureau of Standards Applied Math Series 55 (1964).